JOURNAL OF APPROXIMATION THEORY 2, 389–393 (1969)

Approximation Invariance of Semi-Group Operators under Perturbations

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1. INTRODUCTION

Let X be a real or complex Banach space with norm $\|\cdot\|$, and let $\mathfrak{E}(X)$ be the Banach algebra of endomorphisms of X. Let $\{T(t); t \geqslant 0\}$ be a semi-group of operators of class (C_0) in $\mathfrak{E}(X)$, with infinitesimal generator A. For all λ with $\operatorname{Re}\lambda > \omega_0 = \lim_{t\to\infty}(1/t)\log\|T(t)\|$, the resolvent of A is given by $R(\lambda; A)f = \int_0^\infty e^{-\lambda t}T(t)f\,dt$. One says that $\{T(t)\}$ is *equi-bounded*, if $\|T(t)\| \leqslant M$ $(0 \leqslant t < \infty)$, and *holomorphic*, if $T(t)[X] \subseteq D(A)$ (for all $t > 0$) and $\|AT(t)\| = O(t^{-1})$ $(t \to 0+)$. For all these concepts see, e.g., E. Hille and R. S. Phillips [*7*, Chapters X–XII], and P. L. Butzer and H. Berens [*2*, Chapter I].

There are a number of well-known perturbation theorems for semi-groups. These can be divided essentially into two different types.

If A is the infinitesimal generator of a semi-group $\{T(t)\}$, then one type is concerned, roughly, with conditions upon an operator B, in order that the sum $A + B$ (or the closure of $A + B$) be likewise, the infinitesimal generator of a semi-group. For theorems of this type we refer to, e.g., K. E. Gustafson [*5*], E. Hille and R. S. Phillips [*7*, Chapter XIII], T. Kato [*8*, Chapter IX], V. V. Kucerenko [*9*], I. Miyadera [*10*], H. F. Trotter [*11*] and K. Yosida [*13*]. Some of these theorems are also given for holomorphic semi-groups.

Theorems of the other type state under which conditions upon B the multiplicative perturbation BA likewise generates a semi-group; for these, see, e.g., J. R. Dorroh [*4*], K. E. Gustafson [*6*] and C. F. Widger [*12*].

On the other hand, approximation theorems for semi-groups of operators have been studied under various points of view; see P. L. Butzer and H. Berens [*2*]. A particular case of one of the basic results is [*2*, pp. 88–90]:

Let $\{T(t)\}$ be of class (C_0) and X reflexive. Then the following assertions are equivalent:

(i) $\|T(t)f - f\| = O(t) \qquad (t \to 0+)$,

(ii) $f \in D(A)$.

[1] Corollary 1 answers a question raised by the audience in a colloquium lecture held by the first-named author at Harvard University on November 10, 1966. Part of the results were presented by the second-named author in a talk at the VIIth Austrian Mathematical Congress, Linz, on September 19, 1968.

If $\|T(t)f-f\| = o(t)$ $(t \to 0+)$, *then* $T(t)f \equiv f$ *for all* $t \geq 0$, *whether or not* X *is reflexive.*

The object of this note is to compare the approximation by a perturbed semi-group $\{T'(t)\}$ with that by the unperturbed semi-group $\{T(t)\}$. More specifically, the order of magnitude of $\|T'(t)f-f\|$ as a function of t, is to be compared with that of $\|T(t)f-f\|$. The problem will actually be treated in the setting of the theory of intermediate spaces. A similar question is raised for the resolvent operator $\lambda R(\lambda; A)$ as a function of λ, for $\lambda \to \infty$.

2. Invariance Theorems

In case of an equi-bounded semi-group of class (C_0), we introduce the following subspaces of X (cf. P. L. Butzer and H. Berens [2, Chapter III):

$$X_{\alpha, r; q}(T) = \begin{cases} \left\{ f \in X; \int_0^\infty (t^{-\alpha} \|[T(t) - I]^r f\|)^q \frac{dt}{t} < \infty \right\} \\ \qquad\qquad (0 < \alpha < r;\ 1 \leq q < \infty), \\ \left\{ f \in X;\ \sup_{0 \leq t < \infty} (t^{-\alpha} \|[T(t) - I]^r f\|) < \infty \right\} \\ \qquad\qquad (0 \leq \alpha \leq r; q = \infty), \end{cases}$$

and also

$$\tilde{X}_{\alpha, r; q}(T) = \begin{cases} \left\{ f \in X; \int_0^\infty (t^{r-\alpha} \|A^r T(t) f\|)^q \frac{dt}{t} < \infty \right\} \\ \qquad\qquad (0 < \alpha < r;\ 1 \leq q < \infty), \\ \left\{ f \in X;\ \sup_{0 \leq t < \infty} (t^{r-\alpha} \|A^r T(t) f\|) < \infty \right\} \\ \qquad\qquad (0 \leq \alpha \leq r;\ q = \infty) \end{cases}$$

if $\{T(t)\}$ is holomorphic (r being any fixed positive integer). Our main therorem reads as follows.

THEOREM 1. *Let* $\{T(t)\}$ *and* $\{T'(t)\}$ *be any two equi-bounded semi-groups of class* (C_0). *If* $D(A^r) \subseteq D(A'^r)$, *then* $f \in X_{\alpha, r; q}(T)$ *implies* $f \in X_{\alpha, r; q}(T')$. *If* $\{T(t)\}$ *and* $\{T'(t)\}$ *are, in addition, holomorphic, then* f *belongs to* $\tilde{X}_{\alpha, r; q}(T')$ *if it belongs to* $\tilde{X}_{\alpha, r; q}(T)$.

Remark. This theorem is rather general, since we need not distinguish one semi-group as perturbed and the other as unperturbed, and no hypothesis concerning A and A' is made, other than $D(A^r) \subseteq D(A'^r)$. The latter condition is, for instance, satisfied, in case of both types of perturbation theorems cited in the introduction.

Proof. Since A^r and A'^r are closed operators, the closed-graph theorem yields that A'^r is relatively bounded with respect to A^r, i.e., there exists a constant $C > 0$ such that for all $f \in D(A^r)$,

$$\|A'^r f\| \leqslant C[\|f\| + \|A^r f\|].$$

With the notation $\|f\|_{D(A^r)} = \|f\| + \|A^r f\|$ (similarly for $D(A'^r)$), it follows that for all $f \in D(A^r)$,

$$\|f\|_{D(A'^r)} \leqslant (C+1)\|f\|_{D(A^r)}. \tag{*}$$

Next we consider for $0 < t < \infty$ and every $f \in X$, the function norms (P. L. Butzer and H. Berens [2, pp. 166 ff.]).

$$K(t,f) = K(t,f; X, D(A^r)) = \inf_{f=f_1+f_2} (\|f_1\| + t\|f_2\|_{D(A^r)})$$

and $K'(t,f) = K(t,f; X, D(A'^r))$. By (*), these satisfy the inequality $K'(t,f) \leqslant (C+1)K(t,f)$ $(0 < t < \infty)$. Thus,

$$(X, D(A'^r))_{\theta, q; K} \subseteq (X, D(A^r))_{\theta, q; K},$$

where

$$(X, D(A^r))_{\theta, q; K} = \left\{ f \in X; \int_0^\infty [t^{-\theta} K(t,f)]^q \frac{dt}{t} < +\infty \right\}$$
$$(0 < \theta < 1, 1 \leqslant q < \infty \text{ and/or } 0 \leqslant \theta \leqslant 1, q = \infty).$$

These spaces are intermediate spaces of X and $D(A^r)$ under the obvious norm, i.e., Banach spaces with the property $D(A^r) \subseteq (X, D(A^r))_{\theta, q; K} \subseteq X$. Using the basic equivalence theorem, stating that the spaces $X_{\alpha, r; q}(T)$ are equal to the spaces $(X, D(A^r))_{\alpha/r, q; K}$ $(0 < \alpha < r, 1 \leqslant q \leqslant \infty, r = 1, 2, \ldots)$ and that $X_{r, r; \infty}(T)$ is equal to the space $(X, D(A^r))_{1, \infty; K}$ (see P. L. Butzer and H. Berens [2, pp. 192–193]), we conclude the first part of the theorem.

Concerning the holomorphic case, the basic result (see P. L. Butzer and H. Berens [2, pp. 207 ff.]) that the spaces $X_{\alpha, r; q}(T)$ are equal to the spaces $\tilde{X}_{\alpha, r; q}(T)$ $(0 < \alpha < r,\ 1 \leqslant q \leqslant \infty$ and/or $\alpha = r, q = \infty)$ yields the second part.

Let us now consider the case $q = \infty$, $r = 1$ of Theorem 1. Since, by a simple transformation, each semi-group can be made equi-bounded while remaining in the same class, we obtain

COROLLARY 1. *Let $\{T(t)\}$ and $\{T'(t)\}$ be any two semi-groups of class (C_0) such that $D(A) = D(A')$. For $f \in X$, the following are equivalent:*

(i) $\|T(t)f - f\| = O(t^\alpha)$,

(ii) $\|T'(t)f - f\| = O(t^\alpha) \qquad (t \to 0+; 0 < \alpha \leqslant 1)$.

If, in addition, both semi-groups are holomorphic, then (i) *and* (ii) *are also equivalent to*

$$\text{(iii)} \quad \|AT(t)f\| = O(t^{\alpha-1}),$$
$$\text{(iv)} \quad \|A'T'(t)f\| = O(t^{\alpha-1}) \qquad (t \to 0+;\, 0 < \alpha \leqslant 1).$$

We note that one can also prove the corollary without using the theory of intermediate spaces, by employing the classical perturbation theorems given in E. Hille and R. S. Phillips [*7*; Theorems 13.4.1, Cor. 1 and 13.7.1].

Next, considering the resolvent operator, we have:

THEOREM 2. *Let* $\{T(t)\}$ *and* $\{T'(t)\}$ *be any two equi-bounded semi-groups of class* (C_0). *If* $D(A) \subseteq D(A')$, *then* $\|[\lambda R(\lambda; A)f - f]\| = O(\lambda^{-\alpha})$ *for* $f \in X$, $0 < \alpha \leqslant 1$, *implies* $\|[\lambda R(\lambda; A')f - f]\| = O(\lambda^{-\alpha})$ $(\lambda \to \infty)$.

The proof follows from Theorem 1 and the following one, due to H. Berens [*1*, Chapter 4]: Under the hypothesis of Theorem 2, $\|T(t)f - f\| = O(t^{\alpha})$ $(t \to 0+)$ for $0 < \alpha \leqslant 1$, if and only if $\|\lambda R(\lambda; A)f - f\| = O(\lambda^{-\alpha})$ $(\lambda \to \infty)$.

Concerning results in which the O-condition is replaced by an o-condition, let us point out that such may be established, when further restrictions are imposed upon A as well as on A'. Finally, we emphasize that our results allow many applications, in particular to the initial-value behaviour of solutions of abstract Cauchy-problems. These applications as well as further results, will be published in another paper.

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